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Orthogonality Properties of Iterated Polynomial Mappings

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Abstract. We consider a measure defined on a complex contour and its associated orthogonal polynomials. The action of a polynomial transformation on the measure and the transformation laws of the corresponding orthogonal polynomials are given. Iterating the transformation provides an invariant measure, whose support is the Julia set corresponding to the polynomial transformation. It appears that, up to a constant, the iterated polynomials generated by the initial mapping form a subset of the invariant set of orthogonal polynomials, which fulfill a three term recursion relation. An algorithm is given to compute the coefficients of this recursion relation, which can be interpreted as a linear extension of the iterative procedure.

I. Introduction

The connection between dynamical systems and iterative maps has received a great deal of attention in the past ten years, initiating a new approach to turbulence [1]. The present situation is best summarized in a review article by Eckmann [2]. In particular one dimensional mappings display intriguing properties under iteration [3], as was already noticed by May [4]. Universality properties of one dimensional mappings have been emphasized by Feigenbaum [5], and the corresponding effects observed in experiments [6]. Although more general mappings display also some universality properties [7], the most useful transformations are those who behave in a regular way in the vicinity of their critical point. Special examples of this situation are provided by polynomial mappings.

On the other hand, iterations of rational mappings have been considered sixty years ago by Julia [8] and Fatou [9], and studied extensively by Brolin [10]. The quadratic polynomials have been considered by Myrberg [11], and recently by Douady and Hubbard [12]. A complete understanding of the iterations of a polynomial mapping seems to require the extension from real to complex analysis. There is in fact a relation between mappings and invariant measures [13], and it is therefore interesting to study polynomial transformations on measures.