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Small Random Perturbations of Dynamical Systems and the Definition of Attractors

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Abstract. The "strange attractors" plotted by computers and seen in physical experiments do not necessarily have an open basin of attraction. In view of this we study a new definition of attractors based on ideas of Conley. We argue that the attractors observed in the presence of small random perturbations correspond to this new definition.

1. Introduction

Let (f^t) be a dynamical system, i.e., a group or semigroup of maps $M \mapsto M$ parametrized by a discrete or continuous time t. We assume that M has a topological or differentiable structure, and that the f^t are continuous or differentiable. There often exist subsets Λ of M which attract neighboring points x, this means that $f^t x$ tends to Λ when $t \to \infty$. Such subsets Λ are called attracting sets or attractors. In the simplest cases Λ is an attracting fixed point or periodic orbit. More complicated situations have however been studied, notably Smale's Axiom A attractors (see Smale [35], Bowen [4], and Williams [37]).

Attractors are of interest for the description of the asymptotic behavior of physical systems (or the long term behavior of all kinds of natural phenomena). In particular, some attractors (now called strange attractors) show the phenomenon of sensitive dependence on initial condition, i.e., a small change in initial condition grows exponentially with time (the perturbation need grow only as long as it is small). Landau [20] and Hopf [16] used quasiperiodic attractors[1] to try to describe hydrodynamic turbulence (these attractors do not exhibit sensitive dependence on initial condition). Lorenz [23] found an attractor with sensitive dependence in approximate convection equations, and suggested that this may explain the difficulty of long term weather predictions. Ruelle and Takens [33] proposed that hydrodynamic turbulence is described by non-quasiperiodic attrac-

1 A quasiperiodic attractor is an attracting torus T^m such that the time evolution restricted to T^m becomes, in suitable coordinates, a translation with dense orbits

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