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Unitary Representations of some Infinite Dimensional Groups

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Abstract. We construct projective unitary representations of (a) Map$(S^1;G)$, the group of smooth maps from the circle into a compact Lie group G, and (b) the group of diffeomorphisms of the circle. We show that a class of representations of Map$(S^1;T)$, where T is a maximal torus of G, can be extended to representations of Map$(S^1;G)$,

Introduction

One object of this paper is to describe a series of projective unitary representations of the group of (orientation preserving) diffeomorphisms of the circle. They are characterized, and distinguished from other known representations ([8], [13]), by the property of having "positive energy", which means that the rotation of the circle through an angle α is represented by $e^{-i\alpha K}$ where K is a positive operator.

In their infinitesimal form, i.e. as representations of Vect(S^1), the Lie algebra of smooth vector fields on the circle, the representations have been known for some time to physicists ([5], [3]) in connection with the quantization of strings moving relativistically. (Vect(S^1) is called by physicists the Virasoro algebra.) I have tried to explain briefly in an appendix to this paper how the representations are relevant to the theory of strings; but as a crude oversimplification one can say that one wants to describe unparametrized strings but finds it more convenient to describe parametrized strings: the group of diffeomorphisms acts on the Hilbert space of states of a parametrized string by changing parametrization.

The infinitesimal version of the representations has also been described by Kač([7][7a]).

My approach to the construction of the representations involves constructing irreducible representations of another family of groups. For any Lie group G the group Diff(S^1) of orientation preserving diffeomorphisms of the circle S^1 is a group of automorphisms of the group Map$(S^1;G)$ of smooth maps from S^1 to G (under pointwise composition). Taking first $G=\mathbb{T}$, the circle group, I shall construct an irreducible projective unitary representation of Map$(S^1;\mathbb{T})$ on a Hilbert space H. Then I shall show, what seems to me rather surprising, that any representation

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