Commun. Math. Phys. 79, 33–46 (1981)

Communications in Mathematical Physics

Dirac and Klein–Gordon Equations: Convergence of Solutions in the Nonrelativistic Limit

Ross J. Cirincione and Paul R. Chernoff*

Department of Mathematics, University of California, Berkeley, CA 94720, USA

Abstract. The convergence of solutions of the Dirac and Klein–Gordon equations to solutions of the Pauli and Schrödinger equations in the non-relativistic limit is discussed. An abstract theory of these equations is developed which is general enough to allow physical space to be an arbitrary complete Riemannian manifold.

I. Introduction

Our object is to discuss the sense in which solutions of relativistic wave equations approximate solutions of the corresponding non-relativistic Schrödinger equations when the speed of light tends to infinity. We are concerned specifically with the Dirac and Klein–Gordon equations for a particle in an external electromagnetic field. The problem, at least on a formal level, has a long history, but rigorous results have a rather recent provenance. Veselic [11, 12] dealt with spectral convergence and analyticity of eigenfunctions in $1/c$. His results for the Dirac operator were improved and extended by Hunziker [6]. Veselic also treated the Klein–Gordon equation, but only under extremely restrictive conditions on the electric potential, with vanishing magnetic potential. These authors were concerned with resolvent convergence, and they showed that in some sense the resolvent of the relativistic Hamiltonian converges to that of the non-relativistic Hamiltonian as c tends to ∞.

Once one has resolvent convergence, the Trotter–Kato theorem may be used to deduce convergence of the corresponding one-parameter groups, and hence convergence of solutions of the equations. We follow a variant of this approach, using a generalization of the Trotter–Kato theorem formulated by Davies [4].

In Sect. 2 we discuss an abstract version of the Dirac equation with relatively bounded potentials, simplifying and also generalizing Hunziker's calculations. In Sect. 3 the abstract theory is applied to some concrete cases, including Dirac operators over curved spaces. In Sect. 4 we outline a parallel discussion of the

* Research partially supported by National Science Foundation grant MCS-77-13070

0010-3616/81/0079/0033/$02.80