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Sensitive Dependence to Initial Conditions for One Dimensional Maps*

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Abstract. This paper studies the iteration of maps of the interval which have negative Schwarzian derivative and one critical point. The maps in this class are classified up to topological equivalence. The equivalence classes of maps which display sensitivity to initial conditions for large sets of initial conditions are characterized.

There has been recent interest in the relationship between the "chaotic" asymptotic behavior of complicated solutions to ordinary differential equations and physically unstable phenomena such as those encountered in fluid flow [21]. This has led to numerical studies of a variety of systems of differential and difference equations which appear to have large sets of initial conditions yielding complicated asymptotic behavior. The mathematical theory of Axiom A "strange attractors" provides a satisfying description of some systems which do have complicated asymptotic behavior, but there is little overlap between the numerical studies referred to above and the class of systems with Axiom A attractors. Perhaps the "Lorenz system" [16], is the only example of an explicit system of equations used in a physical problem for which there is a convincing argument that a large set of its solutions have complicated asymptotic behavior. Nonetheless, the numerical studies of such examples as the "Henon" map [9], the mechanical systems studied by Holmes and Moon [11], the "strange attractor" of Spiegel [26], and the density dependent population models of [8] all provide evidence for the prevalence of complicated solutions in systems near those with homoclinic tangencies. From a practical point of view the distinction between trajectories tending to complicated periodic orbits with very long periods and trajectories with aperiodic asymptotic behavior may be slight, but we would like to understand the extent to which numerical computations of strange attractors reflect

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