Commun. Math. Phys. 69, 1–17 (1979)

Communications in
Mathematical
Physics

Invariant Measures for Markov Maps of the Interval*

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Abstract. There is a theorem in ergodic theory which gives three conditions sufficient for a piecewise smooth mapping on the interval to admit a finite invariant ergodic measure equivalent to Lebesgue. When the hypotheses fail in certain ways, this work shows that the same conclusion can still be gotten by applying the theorem mentioned to another transformation related to the original one by the method of inducing.

It is often difficult to decide whether a given map $f: I \to I$ of an interval admits an invariant measure equivalent to Lebesgue measure. Piecewise differentiable f which are expanding [i.e., $\inf |(f^n)'(x)| > 1$ for some n] have such measures under mild additional hypotheses [1, 11, 8, 13, 16]. This paper gives sufficient conditions for certain nonexpandings maps to have invariant measures. This result unifies a number of examples and its conditions are quite computable.

A map $f: I \to I$ of the interval $I = [a, b]$ is *Markov* if one can find a finite or countable collection $\{I_k\}$ of disjoint open intervals such that

a) f is defined on $\cup I_k$ and $I \backslash \cup I_k$ has measure zero.

b) $f|I_k$ is strictly monotonic and extends to a C^2 function on $\bar{I}_k$ for each k,

c) if $f(I_k) \cap I_j \neq \emptyset$, then $f(I_k) \supset I_j$, and

d) there is an R so that $\bigcup_{n=1}^{R} f^n(I_k) \supset I_j$ for every k and j.

A measure μ on I which is equivalent to Lebesgue has the form $\mu(E) = \int_E p(x)dx$ where $p(x)$ is a positive measurable function. We will be trying to understand and apply the following result of Adler [1, 2][1].

Adler's Theorem. *Let* $f: I \to I$ *be Markov,* $M = \sup_{I_k} \sup_{y, z \in I_k} \left| \frac{f''(z)}{f'(y)^2} \right| < +\infty$ *and* $\lambda_n = \inf_x |(f^n)'(x)| > 1$ *for some* n. *Then* f *admits an invariant finite measure* $d\mu = p(x)dx$ *with* $p(x)$ *bounded away from* 0 *and* $+\infty$.