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On a New Derivation of the Navier-Stokes Equation

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Abstract. The Navier-Stokes equation is derived by 'adding' the effect of the Brownian motion to the Euler equation. This is an example suggesting the 'equation': 'Reversible phenomena' ⊕ 'Probability' = 'Irreversible phenomena'.

§1. Introduction

As a model equation representing the motion of viscous incompressible fluid (resp. incompressible perfect fluid), the Navier-Stokes equation (resp. the Euler equation) in a domain $D \subset R^n$ was introduced physically in the 19th century. They are formulated as below:

$$(N.S.)\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } Q_T = (0,T) \times D, \\ \operatorname{div} u = 0 & \text{in } Q_T, \\ u|_{\partial D} = 0 & \text{on } (0,T) \times \partial D, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

and

$$(E)\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = f & \text{in } Q_T, \\ \operatorname{div} u = 0 & \text{in } Q_T, \\ u \cdot \mathbb{n} = 0 & \text{on } (0,T) \times \partial D, \\ u(0,\cdot) = u_0(\cdot), \end{cases}$$

Here $u = u(t,x) = (u^1(t,x), \dots, u^n(t,x))$ denotes the unknown velocity field at a point $(t,x) \in Q_T, p = p(t,x)$ is the unknown pressure at $(t,x), \Delta = \sum_{j=1}^{n} \frac{\partial^2}{(\partial x^j)^2}, \Delta p = \left(\frac{\partial p}{\partial x^1}, \dots, \frac{\partial p}{\partial x^n} \right)$, $\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u^i}{\partial x^i}$, the i-th component of $(u \cdot \nabla) u = \sum_{j=1}^{n} u^j \frac{\partial u^i}{\partial x^j}$, $\mathbb{n}$ is the unit exterior normal, $u_0 = u_0(x) = (u_0^1(x), \dots, u_0^n(x))$ is the given initial velocity