Commun. math. Phys. 63, 257—264 (1978)

Communications in
Mathematical
Physics


# Singularity Structure of the Two-Point Function in Quantum Field Theory in Curved Spacetime*

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**Abstract.** In the point-splitting prescription for renormalizing the stress-energy tensor of a scalar field in curved spacetime, it is assumed that the anticommutator expectation value $G(x,x')=\langle\phi(x)\phi(x')+\phi(x')\phi(x)\rangle$ has a singularity of the Hadamard form as $x\to x'$. We prove here that if $G(x,x')$ has the Hadamard singularity structure in an open neighborhood of a Cauchy surface, then it does so everywhere, i.e., Cauchy evolution preserves the Hadamard singularity structure. In particular, in a spacetime which is flat below a Cauchy surface, for the "in" vacuum state $G(x,x')$ is of the Hadamard form everywhere, and thus the point-splitting prescription in this case has been rigorously shown to give meaningful, finite answers.

A great deal of attention has been focused recently on the problem of renormalizing the stress energy tensor, $T_{\mu\nu}$, of a quantum field in curved spacetime. The quantum field operator itself is mathematically well defined as a distribution, but $T_{\mu\nu}$ is formally a product of field operators (i.e., a product of distributions) and hence is ill-defined until further rules are given for how to evaluate it. The evaluation of $T_{\mu\nu}$ is of considerable interest since it governs the back reaction of the quantum field upon the spacetime geometry.

One procedure for evaluating $T_{\mu\nu}$ which has been studied in detail is the point-splitting renormalization method [1–7]. The basic idea of this approach is to regard $T_{\mu\nu}$ initially as a two-point operator-valued distribution, $\mathscr{T}_{\mu\nu}(x,x')$. $\mathscr{T}_{\mu\nu}(x,x')$ is well defined, but the "coincidence limit" $x\to x'$ is singular. As described below, it has been assumed that the singularity structure of $\mathscr{T}_{\mu\nu}(x,x')$ as $x\to x'$ is of a certain local form. (This has been verified in special cases and there are formal arguments suggesting that it holds in general.) The renormalization ansatz is to subtract from $\mathscr{T}_{\mu\nu}(x,x')$ another, locally constructed, distribution which has the

* Supported in part by NSF grants PHY 77-01432 to Texas A&M University and PHY 76-81102 to the University of Chicago, and by the Alfred P. Sloan Foundation
** NSF Predoctoral Fellow
*** Sloan Foundation Fellow

0010-3616/78/0063/0257/$01.60