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Some Remarks on the Paper of Callias

R. Bott[1] and R. Seeley[2]

[1] Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138, USA
[2] Department of Mathematics, University of Massachusetts, Dorchester, Massachusetts 02125, USA

Abstract. This paper discusses the topological setting of the preceding paper by Callias. In particular, an alternate way of deriving his results is outlined.

1.

The preceding paper brings a nice proof, along new lines, of the index theorem for a special class of elliptic operators on $\mathbb{R}^n$.

The purpose of this note is to point out other methods which could have been used to get at the same result, and at the same time to explain the topological setting of Callias's formula.

We first recall the analytic conditions guaranteeing that the operators considered by Callias are Fredholm, hence have finite index. (Sources are discussed in a remark at the end of this note.) Consider the space

$$\Gamma = \Gamma(\mathbb{R}^n; V)$$

of V-valued smooth functions on $\mathbb{R}^n$, where V is a finite dimensional vector space over $\mathbb{C}$. A differential operator on Γ then has the form

$$D = \sum_{|\alpha| \leqq m} a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha, \tag{1.1}$$

where the $a_\alpha(x)$ are smooth functions from $\mathbb{R}^n$ to the endomorphisms of V. It is assumed that the derivatives of a_α decay at ∞:

$$D^\beta a_\alpha(x) = 0(|x|^{-|\beta|}) \quad \text{as} \quad x \to \infty. \tag{1.2}$$

Then the D in (1.1) induces a *Fredholm operator*

$$D : H_m(\mathbb{R}^n; V) \to L^2(\mathbb{R}^n; V)$$

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