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Separable Coordinates for Four-Dimensional Riemannian Spaces

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Abstract. We present a complete list of all separable coordinate systems for the equations $\sum_{i,j=1}^{4} g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j\Phi)=E\Phi$ and $\sum_{i,j=1}^{4} g^{ij}\partial_i W\partial_j W=E$ with special emphasis on nonorthogonal coordinates. Applications to general relativity theory are indicated.

1. Introduction

We study the problem of separation of variables for the equations

$$\begin{aligned} &\text{a)}\quad \Delta_4\Phi=\sum_{i,j=1}^{4}\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij}\partial_j\Phi)=E\Phi\\ &\text{b)}\quad \sum_{i,j=1}^{4} g^{ij}\partial_i W\partial_j W=E. \end{aligned}\tag{1.1}$$

Here, $ds^2=\sum g_{ij}dx^i dx^j$ is a complex Riemannian metric, $g=\det(g_{ij})$, $\sum_j g^{ij}g_{jk}=\delta^i_k$, $g_{ij}=g_{ji}$, and E is a nonzero complex constant. (Furthermore, we have adopted the notation $\partial_i W=W_i=\frac{\partial W}{\partial x^i}$.) Thus (1.1a) is the Helmholtz equation on a four dimensional complex Riemannian space and (1.1b) is the associated Hamilton-Jacobi (HJ) equation.

In this paper we classify all metrics and coordinate systems for which Equations (1.1) admit solutions via separation of variables. For (1.1a) the separation is in terms of a product whereas for (1.1b) it is in terms of a sum

$$\Phi(\mathbf{x})=\prod_{i=1}\Phi^{(i)}(x^i),\qquad W(\mathbf{x})=\sum_{i=1}^{4}W^{(i)}(x^i).\tag{1.2}$$

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