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On the Pairing of Polarizations

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Abstract. If F is a positive Lagrangian sub-bundle of a symplectic vector bundle (E, ω) we show by elementary means that the Chern classes of F are determined by ω. The notions of metaplectic structure for (E, ω), metalinear structure for and square root of K^F, the canonical bundle of F are shown to be essentially the same. If F and G are two positive Lagrangian sub-bundles with $F \cap \bar{G} = D^{\mathbb{C}}$, we define a pairing of K^F and K^G into the bundle $\mathscr{D}^{-2}(D)$ of densities of order -2 on D. This is the square of Blattner's half-form pairing and so characterizes the latter up to a sign.

Introduction

In order to construct a Hilbert space in the theory of geometric quantization [4, 6, 7], Kostant [3] introduced the notion of half-form normal to a positive polarization. If two positive polarizations F and G are such that $F \cap \bar{G} = D^{\mathbb{C}}$ is smooth, Blattner [1] showed the existence of a pairing of the half forms normal to F and G into the densities of order -1 on D.

If $F \cap \bar{G} = 0$, $\beta \in \Gamma K^F$, $\gamma \in \Gamma K^G$, K^F, K^G the canonical bundles of F and G, then $\beta \wedge \bar{\gamma}$ is a non-singular pairing of K^F and K^G into the volumes on X. Dividing by the Liouville volume gives a function $\langle \beta, \gamma \rangle_0$. In the general case where $F \cap \bar{G} = D^{\mathbb{C}}$, we observe that F and G project into $D^\perp/D$ to give Lagrangian sub-bundles F/D, G/D satisfying $F/D \cap \overline{G/D} = 0$. Thus by dividing out the intersection we can reduce to the case where $F \cap \bar{G} = 0$ and use the exterior product to define a pairing. This pairing is shown to be the square of Blattner's half-form pairing. It is often easier to compute this pairing of the canonical bundles and use continuity arguments to deduce properties of the half-form pairing.

Notation. Let V be a vector space over a field $\mathfrak{k}$, $b = (v_1, \ldots, v_r)$ an r-tuple of elements of V and $A = (A_{ij})$ an $r \times s$ matrix over $\mathfrak{k}$ then $b \cdot A$ will denote the s-tuple with j-th entry $\sum_{i=1}^r A_{ij} v_i$. If b_1, b_2 are r- and s-tuples, (b_1, b_2) will denote the $r+s$-tuple obtained in the obvious way. If T is an endomorphism of V and b an r-tuple, Tb will denote the r-tuple obtained by letting T act componentwise.

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