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A Sketch of Lie Superalgebra Theory

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Abstract. This article deals with the structure and representations of Lie superalgebras ($\mathbb{Z}_2$-graded Lie algebras). The central result is a classification of simple Lie superalgebras over $\mathbb{R}$ and $\mathbb{C}$.

Introduction

"Graded Lie algebras have recently become a topic of interest in physics in the context of supergauge symmetries relating particles of different statistics". See the review [22] from which this quotation is taken and where there is a voluminous bibliography. (See also the review [25].)

In this paper an attempt is made to develop Lie superalgebra theory. Lie superalgebras are often called $\mathbb{Z}_2$-graded Lie algebras. We prefer the term "superalgebra" inspired by physicists. In fact, a Lie superalgebra is not a Lie algebra either graded or not.

We call *superalgebra* any $\mathbb{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, i.e. if $a \in A_\alpha$, $b \in A_\beta$, $\alpha, \beta \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, then $ab \in A_{\alpha+\beta}$. A superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with product [,], satisfying the following axioms

$$[a, b] = -(-1)^{\alpha\beta}[b, a]; \quad [a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]],$$
$$a \in G_\alpha, b \in G_\beta,$$

is called *Lie superalgebra*.

Note that these axioms are satisfied by the Whitehead product in homotopy groups. Lie superalgebras arise also in various cohomology theories, e.g. in deformation theory.

In paper [4] Lie superalgebras are initially introduced as Lie algebras of some generalized groups now called formal Lie supergroups. At present, there is a satisfactory theory analogous to Lie theory connecting Lie superalgebras and Lie supergroups, i.e. groups with functions taking value in some Grassmann algebra, [5].