Commun. math. Phys. 48, 53—79 (1976)

Communications in
Mathematical
Physics

On Resonant Non Linearly Coupled Oscillators with Two Equal Frequencies

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Abstract. This paper contains a detailed study of the flow that the classical Hamiltonian

$$H = \tfrac{1}{2}(x_1^2 + y_1^2) + \tfrac{1}{2}(x_2^2 + y_2^2) + \mathcal{O}_3$$

induces in R^4, $\mathcal{O}_3$ representing a convergent power series that begins with a third order term.
In particular the existence and stability of periodic orbits is investigated.

0. Introduction

This paper contains a detailed description of the flow that the classical Hamiltonian (1.1) induces in its phase space R^4. The Hamiltonian describes two harmonic oscillators with equal frequencies that are coupled through a nonlinear force. This force can be quite general. The only requirement is that it derives from a potential that is represented by a convergent power series in the position and momentum-variables of the oscillators.

Our investigation was stimulated by the special case of the Hénon-Heiles Hamiltonian. A detailed study of that special case can be found in Ref. [1]. Ref. [1] also contains a general result about Hamiltonians of the form (1.1), namely: conditions are formulated under which Moser's twist theorem implies the existence of infinitely many invariant tori on each energy surface (compare the theorem on p. 313). As a side result of our investigation it is shown (Section 5) that these conditions cannot be quite correct and a correction is suggested.

Our detailed investigation of the flow that the Hamiltonian (1.1) induces in R^4 also uses as its main tool the Gustavson normal form. Because the symplectic transformations that leave the leading term of the Hamiltonian (1.1) invariant constitute exactly the group $U(2)$, the Gustavson normal form is best viewed as a function over the Lie algebra of that group. We split the Hamiltonian into two parts: the unperturbed or truncated Hamiltonian consisting of the sum of the leading term and of the first nonvanishing term of the Gustavson normal form,