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Convergence of the Diagonal Operator-valued Padé Approximants to the Dyson Expansion

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Abstract. The diagonal operator-valued Padé approximants formed from the Dyson expansion to the Schrödinger time-evolution operator are shown to converge everywhere in the complex plane, except on a certain subset of the real axis.

The purpose of this paper is to outline a proof of the convergence of the diagonal operator-valued Padé approximants (O.P.A.) formed from the Dyson expansion for the non-relativistic time-evolution operator. This will be carried out by showing that the Dyson expansion belongs to a broad class of operator-valued analytic functions for which the diagonal O.P.A.'s converge.

Let $U(t, z)$ be the time-evolution operator which solves the Schrödinger equation,

$$\begin{aligned} i\frac{d}{dt} U(t, z) &= [H_0 - zV(t)]U(t, z) \\ U(0, z) &= I , \end{aligned} \tag{1}$$

where $V(t)$ is a bounded, positive-definite operator for all time t and H_0 is an unperturbed Hamiltonian. It is known [1] that $U(t, z)$ is an entire function of the coupling constant z and that the Dyson expansion [2],

$$U(t, z) = \sum_{l=0}^{\infty} U_l(t) z^l , \tag{2}$$

converges for all complex z. There are two difficulties concerning this expansion. First of all, for z real, the truncated power series is unitary only to some finite order; secondly, the series may be slowly convergent. On the other hand, Padé approximation techniques can be used to resolve both of these difficulties. In general, the Padé approximants to a series, when they do converge, converge more rapidly than the partial sums; and, in the case of the diagonal ($[N/N]$) approximants, they preserve the unitary character of the operator (cf. Zinn-Justin, Ref. [3]).

Operator-valued Padé approximants are defined analogously to scalar Padé approximants: Let $A(z)$ be a bounded operator-valued function analytic in a neighborhood of the origin and having the power series expansion

$$A(z) = \sum_{l=0}^{\infty} A_l z^l .$$