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# A Central Limit Theorem for the Disordered Harmonic Chain

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**Abstract.** Using methods introduced by Furstenberg and Tutubalin we prove a central limit theorem for the amplitudes of plane waves travelling in a semi-infinite isotopically disordered harmonic chain. This theorem is applied to the problem of heat conduction in disordered harmonic chains.

Fourier's law of heat conduction is that the steady state heat current is proportional to the applied temperature gradient. It is observed to be true over a wide range of temperatures in solids of any degree of purity. The ratio of the heat current to the temperature gradient is the thermal conductivity, and at a fixed temperature this is observed to depend only on the properties of the material. The most convincing explanation of a finite conductivity has been given by Peierls [21]. He showed that anharmonic forces play an essential role in establishing an equilibrium distribution of the energy between the various normal modes of the solid. Assuming the validity of the phonon gas picture he showed that this tendency to equilibrium produces a finite thermal conductivity even for pure crystalline solids. A rigorous formulation of these ideas is certainly the only really satisfactory solution of the problem. It is so difficult however that we look for other soluble models which contain a reflective mechanism and in particular to disordered harmonic systems.

The model we use was first studied by Lebowitz [6, 7]. We take a chain of $N$ particles coupled to their nearest neighbours and connected at each end to heat baths. The heat baths are modelled by white noise whose covariance is proportional to the temperature of the bath and a Langevin damping term to represent the ability of a heat bath to absorb energy. If $x_i(t)$ and $m_i$ are the displacement from its equilibrium position and the mass of the $i$th particle the motion of the particles is determined by the equations

$$\begin{aligned} m_1\ddot{x}_1+2x_1-x_2+\lambda m_1\dot{x}_1&=f_1(t) \\ m_j\ddot{x}_j+2x_j-x_{j-1}-x_{j+1}&=0 \qquad 1<j<N \\ m_N\ddot{x}_N+2x_N-x_{N-1}+\lambda m_N\dot{x}_N&=f_N(t) \end{aligned} \tag{1}$$

where $f_a$ $(a=1, N)$ is a Gaussian random process with variance $2T_a\lambda m_a$. Casher and Lebowitz show that in the steady state situation there is a steady flow of