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A C^*-Algebra of the Two-dimensional Ising Model

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Abstract. We consider the two-dimensional Ising model and show how correlation functions are determined by a state of a C^*-Clifford algebra. We describe how the phase transition manifests itself in terms of a jump in the index of a Fredholm operator. A connection with the Pfaffian approach is made through the theory of unitary dilations of contraction semigroups.

§ 1. Introduction

The two-dimensional Ising model in zero field has been treated algebraically by many authors, notably Onsager [20], Kaufmann [11], Schultz, Mattis, and Lieb [23], Abraham [1, 2], Abraham and Martin-Löf [3]. They consider an array of spins on a finite lattice, compute correlations using either the Clifford algebra [1, 3, 11] or the Fermi algebra [2, 23] and then pass to the thermodynamic limit. Following Pirogov [22] we consider the Clifford and Fermi algebras associated with the infinite lattice. Other C^*-algebras associated with the Ising model are described by Marinaro and Sewell [16].

We investigate the connection between the Gibbs states of the Ising system and certain states of the Clifford algebra. In this we follow Dobrushin [5] and Landford and Ruelle [12] and regard a Gibbs' state of the infinite system as a family of correlations $\langle\sigma_a\ldots\sigma_{a_n}\rangle$ for finite subsets $\{a_1,\ldots,a_n\}$ of the lattice, σ_a taking on values ± 1. These are obtained as the limit of correlation functions for a sequence of finite sublattices with some prescribed boundary conditions. In particular we denote by $\langle\ldots\rangle^p$, $\langle\ldots\rangle^+$ and $\langle\ldots\rangle^-$ the correlation functions which arise from the periodic, "plus" and "minus" boundary conditions respectively. For a review of boundary conditions and general properties of Ising systems see Gallavotti [7]. The state is translationally invariant if $\langle\sigma_{a_1+a}\ldots\sigma_{a_n+a}\rangle = \langle\sigma_{a_1}\ldots\sigma_{a_n}\rangle$ for all lattice vectors $a\in Z^2$ and all subsets $\{a_1,\ldots,a_n\}$. The set of all translationally invariant equilibrium states is a non-empty convex space. A phase transition is said to occur at inverse temperature β_c if for $\beta>\beta_c$ there is more than one equilibrium state while for $\beta<\beta_c$ a unique state exists. Extending a result of Gallavotti and Miracle-Sole [8], Messager and Miracle-Sole [17] have shown that every translationally invariant equilibrium state $\langle\cdot\rangle$ is such that

$$\langle\cdot\rangle=\alpha\langle\cdot\rangle^+ +(1-\alpha)\langle\cdot\rangle^- \quad \text{for} \quad \text{some} \quad \alpha\in[0,1]\,. \tag{1}$$