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Unbounded Derivations of C*-Algebras

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Abstract. We study unbounded derivations of C^*-algebras and characterize those which generate one-parameter groups of automorphisms. We also develop a functional calculus for the domains of closed derivations and develop criteria for closeability. Some special C^*-algebras are considered $\mathfrak{B}\mathfrak{C}(\mathfrak{H})$, $\mathfrak{B}(\mathfrak{H})$, UHF algebras, and in this last context we prove the existence of non-closeable derivations.

I. Introduction

A derivation δ of a C^*-algebra $\mathfrak{A}$ is a linear mapping from a dense * subalgebra $D(\delta) \subset \mathfrak{A}$ to a subspace $R(\delta) \subset \mathfrak{A}$ satisfying the two properties

1. $\delta(AB) = \delta(A)B + A\delta(B), \quad A, B \in D(\delta)$,
2. $\delta(A^*) = -\delta(A)^*, \quad A \in D(\delta)$.

$D(\delta)$ is the domain of δ and $R(\delta)$ the range.

If $\mathfrak{A}$ contains an identity element $\mathbb{1}$ we will always assume $\mathbb{1} \in D(\delta)$ and then $\mathbb{1}^2 = \mathbb{1}$ etc. immediately implies that $\delta(\mathbb{1}) = 0$.

It is known that if a derivation is everywhere defined, $D(\delta) = \mathfrak{A}$, then it is bounded (for this and other results on bounded derivations see, for example, [1], Chapter 4). We will be interested in unbounded derivations. Some results are already given in [2, 3].

II. General Algebras

The principal interest of unbounded derivations is that they arise as infinitesimal generators of strongly continuous one-parameter groups of *-automorphisms of $\mathfrak{A}$.

Let $A \in \mathfrak{A} \mapsto \tau_t(A) \in \mathfrak{A}$ be a one-parameter group of *-automorphisms of the C^* algebra $\mathfrak{A}$ satisfying

$$\lim_{t \to 0} \|\tau_t(A) - A\| = 0, \qquad A \in \mathfrak{A}$$

and define

$$i\delta(A) = \lim_{t \to 0} (\tau_t(A) - A)/t$$

for the set $D(\delta)$ of $A \in \mathfrak{A}$ such that the limit exists. It is easily checked that δ is a derivation of $\mathfrak{A}$ and of course it corresponds to the infinitesimal generator of τ.

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