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Some Operator Algebras in Nested Hilbert Spaces

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Abstract. We prove the existence of a von Neumann algebra of operators and hence the existence of projections acting in any nested Hilbert space. Some other algebras of operators are studied. All those algebras are exhibited in a particular class of Nested Hilbert Spaces, namely sequence spaces.

1. Introduction

Some time ago, Grossmann [1] introduced a generalization of Hilbert space, called a *nested Hilbert space*. In a recent series of papers by the same author and Antoine [2–4], that concept itself was extended further, thus leading to a structure called a *partial inner product space* (PIP-space). Besides the mathematical interest of this object by itself, the aim of that work was to determine the most general framework suitable for the formulation of Quantum Theories. However, for carrying out this reformulation, it is not sufficient to exhibit a larger space of states; one should extend also some key theorems of Hilbert space theory such as Gleason's theorem, the spectral theorem for selfadjoint operators, and so on. A common ingredient to all of these is the notion of projection operator. In particular, a fundamental question is: Does a given PIP-space possess sufficiently many projections?

A suitable definition of projection operators in a PIP-space was given in [3]. The main result is that the usual one-to-one correspondance between projections and appropriately defined subspaces (the so-called PIP-subspaces) still holds. However, there exists PIP-spaces which contain no non-trivial, infinite dimensional PIP-subspaces, thus no nontrivial projections. In a Hilbert space, a standard way of showing the existence of many projections is to exhibit a von Neumann algebra of operators (which is always generated by its projections). In this paper, we will establish the same result for a large class of PIP-spaces, namely the class of nested Hilbert spaces (NHS) introduced earlier [1]. We shall prove that the set of operators acting in any nested Hilbert space contains a von Neumann algebra of operators. Thus a NHS always contains projections. In all examples that we have studied, that von Neumann algebra and thus those projections are non-trivial.

In order to understand the reason of this restriction to nested Hilbert spaces, we shall look first at the general case.

Essentially a PIP-space is a vector space V together with:

1) A family $\{V_r | r \in I\}$ of vector subspaces of V which covers V (i.e. every $f \in V$ is contained in some V_r); when ordered by inclusion, the family admits an order-reversing involution $V_r \leftrightarrow V_{\bar{r}}$.

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