Commun. math. Phys. 42, 41—63 (1975)

Three Applications to $SO(4)$ Invariant Systems of a Theorem of L. Michel Relating Extremal Points to Invariance Properties

Francesco Pegoraro

Scuola Normale Superiore, Pisa, Italy

Received August 1, 1974

Abstract. We consider a theorem due to Michel [1] which relates the invariance properties in peculiar directions in a linear space on which we represent a Lie group G to the extremal points of an arbitrary smooth G-invariant function.

The group we are interested in is $SO(4)$ and we apply the mathematical results to the following problems:

i) mixed linear Stark Zeeman effect in a hydrogen atom,

ii) perturbation of a finite Robertson-Walker metric,

iii) gas evolutions preserving angular momentum and vorticity.

Introduction

Among the solutions of a theory covariant under the action of a group G there may be peculiar ones which are invariant under a subgroup of G.

For example among the orbits of a mass point in a central, stationary and attractive field the circular ones are invariant under rotations around the axis perpendicular to the plane of the orbit.

As it is well known the orbits of a mass point in a central stationary field are specified, modulo a transformation of the covariance group, by the energy E and by the square of the angular momentum J^2.

Now, if we consider the orbits with a fixed value of J^2

$$E = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} + U(r)$$

we see that the allowed values of E[1] go from $E_{\min}$ to infinity. Moreover $E_{\min}$ corresponds to a circular orbit and vice versa.

This example illustrates a property frequently satisfied by highly symmetric solutions, if suitably normalized. Such solutions are extremal i.e. each invariant smooth function defined on the space of "normalized" solutions has a vanishing differential on them.

In the case of Lie groups Michel [1] has proved a theorem which relates invariance properties to extremal points of the invariant functions.

In this paper we want to apply this theorem to the study of three problems which are invariant under the group $SO(4)$.

---

[1] If $\lim_{r \to 0} r^2 |U(r)| < 1$.