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A Simple Class of $U(N)$ Racah Coefficients and Their Application*

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Abstract. Using permutation group techniques, a general expression is derived for the special class of $U(N)$ Racah coefficients for which the representations $[f^1]$ and $[f^3]$ in the recoupling matrix for $[f^1]\times[f^2]\times[f^3]\rightarrow[f]$ are either both totally antisymmetric or both totally symmetric. For the totally antisymmetric case further specialization gives a simple expression for a $U(N)$ Racah coefficient which is needed in taking the average of the product of operators over the states of an irreducible representation of $U(N)$, where this result can be useful in the study of identical fermion systems by spectral distribution methods.

Introduction

In recent years the Wigner-Racah calculus for the unitary groups $U(N)$, with $N>2$, has been brought to a state of development comparable to that for the angular momentum calculus, especially through the work of Biedenharn, Louck, and coworkers [1–7]. For multiplicity-free Wigner couplings, in particular, algebraic formulae for $U(N)$ Wigner coefficients can generally be read off directly from their diagrammatic pattern calculus [1]. Biedenharn and Louck advocate the view that there is a canonical structure for the $U(N)$ Wigner-Racah calculus which eliminates all free choices in the resolution of the multiplicity problem for the most general Wigner coupling. Except for phase there is therefore no arbitrariness in the definition of a $U(N)$ Wigner or Racah coefficient. For $N>3$, however, explicit algebraic constructions for Wigner couplings involving the most general multiplicity structure have so far been limited to matrix elements of the simplest self-adjoint Wigner operators [2], which transform according to the $U(N)$ irreducible representation [211...10]. Louck and Biedenharn [2] also give the $U(N)$ Racah coefficient for the recoupling matrix for $[f]\times[11...10]\times[10...0]\rightarrow[f]$ in elegantly compact form. In the applications to physical problems $U(N)$ Racah coefficients are often more useful than the Wigner coefficients [8, 9]. Being independent of subgroup labels, Racah coefficients also have a simpler algebraic structure than the Wigner coefficients. Despite this fact general expressions for $U(N)$ Racah coefficients for arbitrary N have so far been limited to a few special cases. Even the Racah coefficients for the recoupling transformations for which all four Wigner couplings in the Racah recoupling process are free of multiplicity can not yet be written down directly from a simple pattern calculus, except for a limited number of special cases. When the $U(N)$ representations $[f^2]$ and $[f^3]$ are both totally symmetric (representations with

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