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On the Integral Representation of States on a C*-Algebra

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Abstract. The purpose of this paper is to give some complements to the various extremal decompositions of states on a C^*-dynamical system i.e. a pair (A, G) where A is a C^*-algebra and G is a group acting on A by $*$-automorphisms. We shall see for instance that the method of decomposition associated with a maximal abelian W^*-algebra does not give all the extremal measures in the general case. We also give the explicit form of the greatest lower bound of all the extremal measures and a certain form of continuity of the decomposition. Finally we characterize various systems in the literature (G-abelian algebras, large systems and quasi-large systems) in terms of the equivalence of different notions of ergodicity.

1. Introduction and Notations

Let A be a C^*-algebra with identity, G a group and τ a representation of G in the $*$-automorphism-group $\mathrm{Aut}^*(A)$ of A; in a number of recent articles, the invariant states of A and their integral representation have been intensively studied under certain conditions (G-abelian algebras, asymptotically abelian systems, large systems, etc... (cf. [7, 8, 10, 12, 13]) and more recently Guichardet and Kastler have studied the integral representation of quasi-invariant states (cf. [8]). These systems have many applications in Physics, particularly in Statistical Mechanics (cf. [8, 12]).

The purpose of this paper is to give some complements to the various extremal decompositions in the general case and to find necessary and sufficient conditions for the uniqueness of the decomposition; we shall see, for instance, that the method of decomposition associated with a maximal abelian W^*-algebra does not give all the extremal measures in the general case; we also give the explicit form of the greatest lower bound of all extremal measures and a certain form of continuity of the decompositions.

Finally we characterize various systems cited above in terms of their ergodic states, we give in particular the converse of a result of Ruelle on G-abelian algebras and the converse of a result of Haag, Kastler, Michel and Nagel on "quasi-large" systems.

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Notations. Throughout this note, we use the following notations: A is a C^*-algebra with identity 1, G is a group, τ is a representation of G into $\mathrm{Aut}^*(A)$,

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