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Equilibrium States of the Two-dimensional Ising Model in the Two-Phase Region

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Abstract. We prove that at zero external field and for any temperature below the critical temperature, all translationally invariant equilibrium states for the two-dimensional Ising ferromagnet, are a convex combination of only two extremal states.

I. Introduction

A number of recent works deal with general properties of the equilibrium states of the Ising model. In particular, the following facts have been established. First, the Ising ferromagnet with an external field $h \neq 0$ has always a unique equilibrium state, as was proved by Ruelle [1] and Lebowitz and Martin-Löf [2]. It was also shown in [2] that there is only one equilibrium state at $h=0$ and above the critical temperature $T \geqq T_c$; T_c being defined as the temperature above which there is no spontaneous magnetization. For $h=0$ and at sufficiently low temperature $T \leqq T_0 < T_c$, Gallavotti and Miracle-Sole [3] have shown that every translationally invariant equilibrium state is a convex linear combination of only two extremal states. This results hold in any dimension $\nu \geqq 2$. Moreover, in the two-dimensional case Lebowitz [4] has proved that the above defined value of T_c coincides with the Onsager value of the critical temperature.

It is the aim of this paper to show that in the two-dimensional case, for $h=0$ and for all values of the temperature below the critical value T_c, every translationally invariant equilibrium state is a convex linear combination of only two extremal equilibrium states. The description of the translationally invariant equilibrium states of the two-dimensional Ising ferromagnet is then complete. This model has two pure phases coexisting when $h=0$ and $T < T_c$, and only one pure phase is present for all other values of h and T.

We recall that, following Dobrushin [5] and Lanford and Ruelle [6], an equilibrium state of the infinite system may be defined as a family of correlation functions $\langle \sigma_{a_1} \ldots \sigma_{a_n} \rangle$ for the finite sets $\{\sigma_{a_1}, \ldots, \sigma_{a_n}\}$ of spins of the lattice, obtained as the limit of correlation functions for a sequence of finite boxes with some boundary conditions. The state is translationally invariant if $\langle \sigma_{a_1+a} \cdots \sigma_{a_n+a} \rangle = \langle \sigma_{a_1} \cdots \sigma_{a_n} \rangle$ for all a of the