Commun. math. Phys. 40, 153—174 (1975)

General Concept of Quantization

F. A. Berezin*

Academy of Sciences of the Ukrainian SSR, Institute for Theoretical Physics, Kiev, Ukrainian, SSR

Received June 15, 1974

Abstract. The general definition of quantization is proposed. As an example two classical systems are considered. For the first of them the phase space is a Lobachevskii plane, for the second one the two-dimensional sphere.

It is generally accepted that the quantization is an algorithm by means of which a quantum system corresponds to a classical dynamic one. Furthermore, it is required that in the limit $h \to 0$ where h is the Planck's constant, a quantum dynamic system change to a corresponding classical one. This requirement is called the correspondence principle. It is quite obvious that there exist quite a lot quantizations obeying the correspondence principle; the quantum description of a physical phenomenon is more detailed than the classical one, and so there are certain phenomena the difference between which is displayed in their quantum description, whereas their classical description does not show this difference.

The following intuitive method of quantizing the classical dynamic systems with a flat phase space has become well known since the Schrödinger equation was first written down. If a system has n degrees of freedom, its phase space is a real linear space $\mathscr{R}^{2n}$ of dimension $2n$, and the observables are the functions $f(p,q)$, $p,q \in \mathscr{R}^{2n}$, $p=(p_1 \ldots p_n)$, $q=(q_1 \ldots q_n)$, where p_i, q_i are the momenta and coordinates. The Hilbert space of states of a corresponding quantum system is a space of functions $f(x)$, $x=(x_1, \ldots, x_n)$ of n real variables with a summable square. The operators in $L^2(\mathscr{R}^n)$ $\hat{p}_k$, $\hat{q}_k$ are compared to the classical momenta and coordinates, p_k, q_k using the formulas

$$(\hat{q}_K f)(x) = x_K f(x), \qquad (\hat{p}_K f)(x) = \frac{h}{i}\frac{\partial f}{\partial x_K}. \tag{0.1}$$

A "quantum observable" – the operator $f(\hat{p},\hat{q})$ obtained by "replacing the real variables p_i, q_i by the operators $\hat{p}_i$, $\hat{q}_i$ in $f(p,q)$" corresponds to an arbitrary observable $f(p,q)$. However, the operators $\hat{p}_i$, $\hat{q}_i$ do not commute, and so this algorithm is applicable, provided that the analytical expression for f contains no products such as $p_i q_i$. For the case of arbitrary $f(p,q)$ the algorithm in question should be specified. One of such specifications which posseses a number of remarkable features is due to Weyl [1].

* The Moscow State University.