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On the Algebra of Test Functions for Field Operators

Decomposition of Linear Functionals into Positive Ones

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Abstract. It is shown that every continuous linear functional on the field algebra can be defined by a vector in the Hilbert space of some representation of the algebra. The functionals which can be written as a difference of positive ones are characterized. By an example it is shown that a positive functional on a subalgebra does not always have an extension to a positive functional on the whole algebra.

1. Introduction

The formulation of the reconstruction theorem of Wightman [1] in terms of positive functionals on the tensor algebra over a space of test functions [2] provides a natural framework for a study of the nonlinear restrictions on the Wightman distributions. This is the reason why the properties of this algebra are of interest and have been the subject of several investigations [2–7]. This is also the motivation for the present paper, although we shall here ignore the linear conditions of field theory and be concerned with the positive linear functionals in general. As a space of test functions we take Schwartz space $\mathscr{S}$ and we denote the algebra by $\underline{\mathscr{S}}$. It is shown that there exist so many positive functionals that the corresponding Hilbert norms define a topology on $\underline{\mathscr{S}}$ which is identical to the usual one. This topology, however, is not well adapted to the order structure on $\underline{\mathscr{S}}$ in the sense that continuous linear functionals need not be of the form $(T_1 - T_2) + i(T_3 - T_4)$ with positive functionals T_i. This is connected with the fact that the multiplication on the algebra is not continuous in both variables jointly. Let τ denote the usual topology on $\underline{\mathscr{S}}$ and $\mathscr{N}$ the strongest convex topology such that the multiplication is a jointly continuous bilinear map $\underline{\mathscr{S}}[\tau] \times \underline{\mathscr{S}}[\tau] \to \underline{\mathscr{S}}[\mathscr{N}]$. It is shown that functionals of the above form are exactly the $\mathscr{N}$-continuous functionals.

Finally we consider the problem of extending a positive functional from a subalgebra of $\underline{\mathscr{S}}$ to a positive functional on the whole algebra. In some cases this is shown to be possible, but an example is also given where no extension is a linear combination of positive functionals.