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On Birkhoff's Theorem in General Relativity

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Abstract. Two generalisations of Birkhoff's theorem are proved for the cases where the three-parameter group of motions acts on two-dimensional time-like and null orbits. A complete list of possible extensions of the three-parameter group to one of four parameters and of the resulting metrics is given.

§ 1. Introduction

In a series of papers Plebanski, Stachel, and Goenner [1–3] have considered space-times admitting three-parameter isometry groups with two-dimensional surfaces of transitivity. Following Goenner [3] we will denote these groups by $G_3(2s)$, $G_3(2t)$ or $G_3(2n)$ when the orbits are space-like, time-like or null respectively. This class of space-times includes the spherically symmetrical case for which the well-known theorem of Birkhoff [4] is valid: a spherically symmetrical vacuum space-time admits a fourth hypersurface orthogonal Killing vector. In this paper the following generalisation of the above theorem will be proved[1]: a space-time admitting a three-parameter group of isometries with two dimensional non-null orbits and with a Ricci tensor of Segré type [(11) (11)] or [(11 11)] admits a fourth hypersurface orthogonal Killing vector. This theorem was proved in [3] for the case of space-like orbits but the analysis for time-like orbits was incorrect. An account of the algebraic classification of the Ricci tensor is given in references [5] and [6]. A vacuum space-time admitting a $G_3(2n)$ is shown to be a plane-wave space-time. In the above the fourth Killing vector commutes with the original three Killing vectors (i.e. the extension of the group $G_3(2)$ is central). It is interesting to investigate the case where the extension is non-central. In § 3 a complete list of extensions of a $G_3(2)$ to a G_4 and of the resulting metrics is given. This list includes the results for $G_3(2n)$ and a number of other cases omitted in [3].

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[1] In fact for one special case mentioned at the end of § 2 (ii) the theorem is not valid.