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On Global Solutions for Non-Linear Hamiltonian Evolution Equations

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Abstract. It is shown that partial differential equations of Hamiltonian type admit global solutions in time if (a) the initial data is near equilibrium (or the coupling constant is small) (b) the linear terms have positive energy and (c) the non-linear terms are smooth functions in the topology of the linearized energy norm. The non-linear terms need not have positive energy. The result is applied to non-linear wave equations in which the interaction energy is not necessarily positive.

1. Introduction

Our purpose is to give a simple technique for proving global existence of solutions for Hamiltonian systems in which the energy is not necessarily positive. Despite the trivial appearance of the theorem, it is applicable to non-linear wave equations in which case we can remove the positivity assumption on the energy which is commonly made. While our results are not as deep as those of other authors; cf. Chadam [1], they have the advantage of extreme simplicity. The methods will be applied elsewhere to study global solutions of the Einstein equations in general relativity cf. [3].

2. Abstract Theorem

To avoid a lot of notation concerning symplectic structures, we shall just introduce Hamiltonian systems in a special form suitable for our present purposes cf. [2].

Let $\mathbb{H}$ be a (real) Hilbert space and A an (unbounded) skew adjoint operator on $\mathbb{H}$. Thus A generates a one parameter group of isometries on $\mathbb{H}$. Let $V:\mathbb{H}\to\mathbb{R}$ be a smooth function and suppose:

$$V(0)=0,\ \ DV(0)=0,\ \ D^2V(0)=0$$

where $DV(0)$ is the derivative of V at 0. (Actually we require only V defined on D_A, the domain of A.)

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