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Note on Trace Inequalities

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Abstract. Under conditions which are sufficiently general for physical applications the trace inequalities

$$\operatorname{tr} e^{-(A+B)} \leqq \operatorname{tr} e^{-A} e^{-B}$$

and

$$|\operatorname{tr} e^{-(A+iB)}| \leqq \operatorname{tr} e^{-A}$$

with A and B self adjoint are shown in a rigorous manner.

Introduction

Recent investigations in statistical mechanics and in particular the treatment of systems of N particles with gravitational and electromagnetic interactions have shown that trace inequalities serve as a prominent tool [1]. The expression for the partition function $z(\beta, \lambda) = \operatorname{tr} e^{-\beta H(\lambda)}$ may serve as an example. Unfortunately the trace operation in an infinitely dimensional space is somewhat more complicated than in its finite dimensional counterpart and there seems to be some confusion about the conditions under which the traces in the general case really exist [2].

While the operators for which a finite trace can be defined – the so-called trace class operators – are well known in the mathematical literature, little can be found concerning trace class operators (or more generally operators with finite p-norm) which are of the form e^{-A}. All that can be said about the operator A is that it *must* be unbounded and that it *must* have a compact resolvent. It is exactly the unboundedness of the exponent which creates all the difficulties with domain questions etc. So our assumptions in the following may seem rather restrictive; they are, however, general enough for the purpose of applications to physical problems. The usual spectral theorems are not adequate because applications like the analyticity of the partition function lead to non-normal operators.

Notation. We denote by $\|A\|_p$ $1 \leqq p < \infty$ the familiar p-norm for compact operators on $\left(\text{i.e. } \|A\|_p = \left(\sum_i a_i^p\right)^{1/p}, a_i \text{ the eigenvalues of } (AA^\dagger)^{1/2}\right)$ and by $\|A\|_\infty$ the usual operator norm. $\mathscr{B}_p$, $1 \leqq p < \infty$, is the space of

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