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Unbounded Functionals on a Group Algebra and Applications to Quantum Theory

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Abstract. A certain class of positive functionals on a group algebra is examined that is pertinent to the induced representations of Frobenius and Mackey. Though these functionals are not bounded in the L^1 norm, continuity still persists to an extent that secures the existence of a continuous group representation obtained from Gelfand's construction. The theory thus developed provides a new aspect of both the "improper states" in quantum theory and the induced representations of groups. The method is applied to the Poincaré group and it is shown that the representations, in which particles can be accommodated, are determined up to unitary equivalence by unbounded functionals of a simple structure. It is stressed that representations describing an infinitely degenerate vacuum emerge from mass nonzero representations as the mass tends to zero.

1. Introduction

Positive definite functionals are of prime interest for the harmonic analysis on groups, and their mathematical treatment has arisen from the conjunction of two different studies. On the one hand we have the work on group representations and on the other hand there is the theory of Banach algebras as advanced by Gelfand, which has now acquired the necessary technical adaptability as a suitable instrument for dealing with locally compact groups. Positive definite functions give rise to positive functionals on the L^1 algebra of the group. However, there are numerous positive functionals, which are unbounded with respect to the L^1 norm and hence do not arise from positive definite functions, yet they give rise to perfectly sound, i.e. continuous unitary, representations of the group. Two examples shall illustrate this point. The first is due to Godement [1].

1. Let G be a locally compact group, $L(G)$ the $*$-algebra of continuous functions on G with compact support, and F the functional on $L(G)$ defined by $F(f)=f(e)$ with e being the identity in G. Then $F(f^*f) = \|f\|_2^2 \geqq 0$. Thus the functional F is positive, but is clearly not bounded in the L^1 norm unless G is discrete. Nevertheless, the operators $U(x)$, $x\in G$, given by $(U(x)f)(y)=f(x^{-1}y)$, are bounded on $L^2(G)$ and define a continuous unitary representation of G, called the left regular representation.