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Einstein Tensor and Generalizations of Birkhoff's Theorem*

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Abstract. The Einstein tensors of metrics having a 3-parameter group of (global) isometries with 2-dimensional non-null orbits $G_3(2, s/t)$ are studied in order to obtain *algebraic* conditions guaranteeing an additional normal Killing vector. It is shown that Einstein spaces with $G_3(2, s/t)$ allow a G_4. A critical review of some of the literature on Birkhoff's theorem and its generalizations is given.

1. Introduction

Birkhoff's theorem is stated, usually, in one of the following forms [1, 2]: (1) The most general spherically symmetric solution of Einstein's vacuum field equations is the Schwarzschild exterior solution [3] and (2) any spherically symmetric solution of Einstein's vacuum field equations is static [4].

A metric is defined to be spherically symmetric if it allows as a group of isometries the rotation group $0(3, R)$ acting on spacelike 2-dimensional orbits. All corresponding Killing vectors are normal (hypersurface orthogonal). The metric is defined to be static if it allows a timelike and normal Killing vector

$$\xi_{(\alpha;\beta)} = 0, \quad \xi_{[\alpha}\xi_{\beta,\gamma]} = 0, \quad \xi^\alpha \xi^\beta g_{\alpha\beta} > 0\,. \tag{1}$$

Conditions (1) are local and, in general, limited to certain coordinate neighborhoods of the Riemannian manifold. For example, the Schwarzschild exterior solution (in Schwarzschild coordinates)

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2}$$

allows a timelike normal Killing vector $\xi^\alpha = \delta^\alpha_0$ only for the range $r > 2m$. Beyond the null surface $r = 2m$, this Killing vector stays normal yet becomes spacelike. As Eq. (1) shows, this behavior is independent of the coordinates chosen.

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