Commun. math. Phys. 10, 274—279 (1968)

An Existence Proof for the Gap Equation in the Superconductivity Theory

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Received May 28, 1968

Abstract. An existence theorem for the "gap equation" in the superconductivity theory is given, as a consequence of the Schauder-Tychonoff theorem. Sufficient conditions on the kernel are given, which insure the existence of a solution amongst a particular class of continuous functions. The case of a positive kernel is studied in detail.

1. Introduction

For a non relativistic many-fermion system the existence of a "superfluid" or "superconducting" state is related to the appearence of non trivial solutions in a non linear integral equation, called the "gap equation".

Various approximation methods for finding the solution of the gap equation have been devised [1, 2, 3], which give rise to a "linearization" of the equation. All these methods produce solutions with the same non-analytic behaviour for small values of the interaction strength. A necessary condition for the appearence of non trivial solutions has been given a long time ago by Cooper, Mills and Sessler [4] (see also ref. 1). The convergence of an iterative procedure has been proved, under certain conditions, by Kitamura [5]. Fixed point theorems were first used by Odeh [6]. We prove here an existence theorem under entirely different assumptions, which cover many cases of physical interest. We make use of the Schauder-Tychonoff theorem, which allows us to find a solution amongst a particular class of continuous functions.

2. The Existence Theorem

Let us consider the gap equation in its simplest form (i.e. the equation for the spherically symmetrical solutions at zero temperature):

$$\varphi(k) = \int_0^\infty K(k, k') \frac{\varphi(k')}{\sqrt{(k'^2 - 1)^2 + \varphi(k')^2}} dk' \tag{1}$$

* Supported by contract D. G. R. S. T. No. 6700 976.