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A Generalization of the Spherical Harmonic Addition Theorem

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Abstract. A generalization of the spherical harmonic addition theorem is proved. The resulting polynomial with four parameters, which corresponds to the Legendre polynomial for the usual spherical harmonic addition theorem, is expressed as four different but equivalent series. Each of them is a finite series of the Gegenbauer polynomials. Thereby the symmetry properties of this polynomial are clarified.

§ 1. Introduction and Summary

In several problems [1] of mathematical physics, there is a need to generalize the spherical harmonic addition theorem. Let us consider the functions of two points Ω and Ω' on a unit sphere,

$$B_{L,M}^{l,l'}(\Omega, \Omega') = \sum_m C(ll'L; m, M-m)\, Y_{lm}(\Omega)\, Y_{l'M-m}(\Omega')\,, \tag{1}$$

which transform by a rotation as the $(2L+1)$ components of a spherical tensor [2] of rank L. In (1), $C(ll'L; m, M-m)$ is a Clebsh-Gordan coefficient (CGC). From two vectors we can construct only $L+1$ or L linear independent spherical tensors or pseudo tensors of rank L. Hence, we can expect and also prove that $B_{L,M}^{l,l'}(\Omega, \Omega')$ can be expanded in the following forms:

$$B_{L,M}^{l,l'}(\Omega, \Omega') = \sum_{s=\max(0, L-l')}^{\min(l,L)} F_{L;s}^{l,l'}(t)\, B_{L,M}^{s,L-s}(\Omega, \Omega') \tag{2}$$

$$\text{for} \quad l + l' - L = \text{even}\,,$$

and

$$B_{L,M}^{l,l'}(\Omega, \Omega') = \sum_{s=\max(1, L-l'+1)}^{\min(l,L)} F_{L;s}^{l,l'}(t)\, B_{L,M}^{s,L-s+1}(\Omega, \Omega'), \tag{2'}$$ [1]

$$\text{for} \quad l + l' - L = \text{odd}\,.$$

In Eqs. (2) and (2′),

$$t = \cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$$

with $\Omega = (\theta, \varphi)$ and $\Omega' = (\theta', \varphi')$, and $F_{L;s}^{l,l'}(t)$ is a rotationally invariant function of t.

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[1] H. Joos used functions slightly different from our $B_{L,M}^{s,L-s+1}(\Omega, \Omega')$ in this case.