Commun. Math. Phys. 183, 477–510 (1997)

Communications in
Mathematical Physics

Classification of N-(Super)-Extended Poincaré Algebras and Bilinear Invariants of the Spinor Representation of $Spin\ (p,q)$

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Received: 4 December 1995 / Accepted: 16 May 1996

Abstract: We classify extended Poincaré Lie super algebras and Lie algebras of any signature (p, q), that is Lie super algebras (resp. Z_2-graded Lie algebras) $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{so}(V)+V$ is the (generalized) Poincaré Lie algebra of the pseudo-Euclidean vector space $V = \mathbb{R}^{p,q}$ of signature (p, q) and $\mathfrak{g}_1 = S$ is the spinor $\mathfrak{so}(V)$-module extended to a $\mathfrak{g}_0$-module with kernel V. The remaining super commutators $\{\mathfrak{g}_1, \mathfrak{g}_1\}$ (respectively, commutators $[\mathfrak{g}_1, \mathfrak{g}_1]$) are defined by an $\mathfrak{so}(V)$-equivariant linear mapping

$$\vee^2\mathfrak{g}_1 \to V \quad \text{(respectively,} \quad \wedge^2\mathfrak{g}_1 \to V).$$

Denote by $\mathcal{P}^+(n, s)$ (respectively, $\mathcal{P}^-(n, s)$) the vector space of all such Lie super algebras (respectively, Lie algebras), where $n = p + q = \dim V$ and $s = p - q$ is the classical signature. The description of $\mathcal{P}^\pm(n, s)$ reduces to the construction of all $\mathfrak{so}(V)$-invariant bilinear forms on S and to the calculation of three $\mathbb{Z}_2$-valued invariants for some of them.

This calculation is based on a simple explicit model of an irreducible Clifford module S for the Clifford algebra $Cl_{p,q}$ of arbitrary signature (p, q). As a result of the classification, we obtain the numbers $L^\pm(n, s) = \dim \mathcal{P}^\pm(n, s)$ of independent Lie super algebras and algebras, which take values 0,1,2,3,4 or 6. Due to Bott periodicity, $L^\pm(n, s)$ may be considered as periodic functions with period 8 in each argument. They are invariant under the group Γ generated by the four reflections with respect to the axes $n = -2$, $n = 2$, $s - 1 = -2$ and $s - 1 = 2$. Moreover, the reflection $(n, s) \to (-n, s)$ with respect to the axis $n = 0$ interchanges L^+ and L^- :

$$L^+(-n, s) = L^-(n, s).$$

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