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Quantum Analysis – Non-Commutative Differential and Integral Calculi

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Abstract: A new scheme of quantum analysis, namely a non-commutative calculus of operator derivatives and integrals is introduced. This treats differentiation of an operator-valued function with respect to the relevant operator in a Banach space. In this new scheme, operator derivatives are expressed in terms of the relevant operator and its inner derivation explicitly. Derivatives of hyperoperators are also defined. Some possible applications of the present calculus to quantum statistical physics are briefly discussed.

I. Introduction

In theoretical sciences, non-commutative operators play an important role. In particular, the differentiation of an operator $A(t)$ with respect to the relevant parameter t is frequently used. As is well known, the derivative of $A(t)$ is defined by

$$A'(t) \equiv \frac{dA(t)}{dt} = \lim_{h\to 0} \frac{A(t+h) - A(t)}{h} . \tag{1.1}$$

Norm convergence of (1.1) can be discussed in a Banach space and strong convergence is appropriate for unbounded linear operators.

In many situations, we treat an operator-valued function $f(A(t))$ of the operator $A(t)$, such as an exponential operator [1–15] $\exp A(t)$. Since the derivative $A'(t)$ does not commute with $A(t)$ in general, the derivative of $\exp A(t)$ is given by the following integral [1, 3, 5]:

$$\frac{d}{dt} e^{A(t)} = e^{A(t)} \int_0^1 e^{-\lambda A(t)} A'(t) e^{\lambda A(t)} \, d\lambda . \tag{1.2}$$

When $A'(t) \equiv dA(t)/dt$ commutes with $A(t)$, we have

$$\frac{d}{dt} e^{A(t)} = e^{A(t)} \frac{dA(t)}{dt} . \tag{1.3}$$