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C*-Algebras Associated With One Dimensional Almost Periodic Tilings

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Abstract: For each irrational number, $0 < \alpha < 1$, we consider the space of one dimensional almost periodic tilings obtained by the projection method using a line of slope α. On this space we put the relation generated by translation and the identification of the "singular pairs." We represent this as a topological space X_α with an equivalence relation R_α. On R_α there is a natural locally Hausdorff topology from which we obtain a topological groupoid with a Haar system. We then construct the C*-algebra of this groupoid and show that it is the irrational rotation C*-algebra, A_α.

Given a topological space X and an equivalence relation R on X, one can form the quotient space X/R and give it the quotient topology. It frequently happens however that the quotient topology has very few open sets. For example let X be the unit circle, which we shall write as [0,1] with the endpoints identified and the group law given by addition modulo 1. Fix α, irrational, $0 < \alpha < 1$, and let $R = \{(x, y) \mid x - y \in \mathbb{Z} + \alpha\mathbb{Z}\}$. Since each equivalence class of R is dense in X, the only open sets in X/R are $\emptyset$ and X/R.

However the equivalence relation R has the structure of a groupoid and if we can put a topology on R, (usually not the product topology of $X \times X$), so that R becomes a topological groupoid:

(i) $R \ni (x,y) \mapsto (y,x) \in R$ is continuous, and
(ii) $R^2 \ni ((x,y),(y,z)) \mapsto (x,z) \in R$ is continuous,

and we can find a compatible family $\{\mu^x\}$ of measures (μ^x is a measure on $R^x = \{(x,y) \mid x \sim y\}$), called a Haar system (see Renault [7, Definition I.2.2]), one can construct a C*-algebra, $C^*(R,\mu)$, by completing $C_{oo}(R)$, the continuous functions on R with compact support in a suitable norm.

In the example above of the relation R on the unit circle S^1, suppose $(x,y) \in R$, so there is $n \in \mathbb{Z}$ such that $(x + n\alpha) - y \in \mathbb{Z}$ and let $\mathcal{U} \subseteq S^1$ be a neighbourhood

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