Commun. Math. Phys. 183, 291–306 (1997)

Communications in Mathematical Physics

Higher Order Terms in the Melvin–Morton Expansion of the Colored Jones Polynomial

L. Rozansky

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, U.S.A.
E-mail: rozansky@math.ias.edu

Received: 15 March 1996 / Accepted: 8 June 1996

Abstract: We formulate a conjecture about the structure of "upper lines" in the expansion of the colored Jones polynomial of a knot in powers of $(q-1)$. The Melvin–Morton conjecture states that the bottom line in this expansion is equal to the inverse Alexander polynomial of the knot. We conjecture that the upper lines are rational functions whose denominators are powers of the Alexander polynomial. We prove this conjecture for torus knots and give experimental evidence that it is also true for other types of knots.

1. Introduction

Ever since the discovery of the Jones polynomial, its relation to the objects of the classical topology, i.e. the fundamental group of a knot, remained somewhat of a mystery. An apparent similarity between the skein relations for the Jones and Alexander polynomials did not lead to a better understanding of this relation. Therefore the discovery by P. Melvin and H. Morton [9] of the inverse Alexander polynomial inside the $(\check{q}-1)$ expansion of the colored Jones polynomial was a very interesting development.

Let $\mathcal{K}$ be a knot in S^3. We denote by $J_\alpha(\mathcal{K};K)$ its colored Jones polynomial normalized in such a way that it is multiplicative under a disconnected sum and

$$J_\alpha(\text{unknot};K) = \frac{\sin(\frac{\pi}{K}\alpha)}{\sin(\frac{\pi}{K})} = \frac{\check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}}}{\check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}}}, \quad \check{q} = e^{\frac{2\pi i}{K}}\,. \tag{1.1}$$

Another popular normalization for the Jones polynomial is

$$V_\alpha(\mathcal{K};K) = \frac{J_\alpha(\mathcal{K};K)}{J_\alpha(\text{unknot};K)}, \quad V_\alpha \in \mathbb{Z}[\check{q},\check{q}^{-1}]\,. \tag{1.2}$$

For a fixed value of color α we can expand the Jones polynomial $V_\alpha(\mathcal{K};K)$ in Taylor series in powers of

$$h = \check{q} - 1\,, \tag{1.3}$$