Commun Math Phys 182, 505–533 (1996)

Communications in Mathematical Physics

The Atiyah–Singer Index Theorem as Passage to the Classical Limit in Quantum Mechanics

George A. Elliott[1,2], Toshikazu Natsume[3], Ryszard Nest[1]

[1] Mathematics Institute, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
[2] Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 1A1
[3] Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14214, USA

Received: 11 August 1995 / Accepted: 5 December 1995

Abstract: A new approach to the Atiyah–Singer index theorem is described, using the technique of continuous fields of C^*-algebras. The proof is given in the case of elliptic pseudodifferential operators on $\mathbb{R}^n$.

1. Introduction

In the study of pseudodifferential operators (abbreviated ΨDOs) on a Euclidean space, or, more generally, on an open manifold, in order to extract information on the global properties of a ΨDO, one imposes "boundary" conditions. For instance, Seeley [S] investigated the class of elliptic ΨDOs of order 0 on $\mathbb{R}^n$ with the property, in a certain sense, of being equal to the identity at infinity. If P belongs to this class, then its kernel and cokernel are finite-dimensional, and hence the (analytical) index,

$$\operatorname{index} P = \dim \operatorname{Ker} P - \dim \operatorname{Coker} P ,$$

is defined. For such operators, an index formula of Atiyah–Singer type has been established (see, for instance, [B-B]). Hörmander [H] also studied a class of ΨDOs of order 0 on $\mathbb{R}^n$ and obtained an index formula. Bott and Seeley [B-S] and Callias [Ca] studied differential operators with coefficients and their derivatives decaying at infinity.

The classes of ΨDOs mentioned above cover a wide variety of operators. Nevertheless, interesting operators are left out of the picture (or at least not directly handled). For instance, in the one-dimensional case, the operator $D = x + \frac{d}{dx}$ is one of them. Not to mention that D is an important operator in quantum mechanics.

In [E-N-N] we presented a new, simple, proof of the Connes Isomorphism Theorem for C^*-algebra crossed products by $\mathbb{R}$-actions, using continuous fields of C^*-algebras. As a byproduct, we obtained an index formula for the operator D ([E-N-N, Remark 4.8]). In the present paper, we study a whole class of ΨDOs by similar methods, and establish an index formula (Theorem 3.1), describing the index in terms of the symbol.

Let us describe briefly the content of this paper. The general idea behind our proof is to consider the elements of $C_0(T^*\mathbb{R}^n)$ as the classical limit (as $\hbar$ goes to