Commun. Math. Phys. 181, 587 – 603 (1996)

Communications in Mathematical Physics

The Toda Hierarchy and the KdV Hierarchy

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Received: 5 February 1996 / Accepted: 22 March 1996

Abstract: This paper considers the relation between the periodic KdV hierarchy and the limit of the periodic Toda hierarchies. By choosing the initial data of the Toda flows in a canonical way, the behavior of a certain Toda flow can mimic KdV flows. Conjecturally, a method of deforming the KdV hierarchy is given.

McKean and Trubowitz [2] showed that the theory of the KdV equation

$$\frac{\partial}{\partial t}g(x,t) = \frac{\partial^3}{\partial x^3}g(x,t) - 6g(x,t)\frac{\partial g}{\partial x}(x,t)$$

is intimately related to the geometry of a related hyperelliptic curve of infinite genus, the Bloch spectrum $\mathscr{B}_{g_t}$ of the operator

$$L_{g_t} : \psi \to \frac{d^2}{dx^2}\psi(x) + g(x,t)\psi(x)\,,$$

where $g_t = g(x,t)$. As was known classically, $\mathscr{B}_{g_t}$ is independent of t, when $g(x,t)$ evolves according to the KdV equation. Our purpose in this paper is to develop a theory of finite difference operators and their Bloch spectra and isospectral flows which mimics the KdV theory. The basic idea of this paper is to use the theory of the periodic Toda chain of length N. Here again, the periodic Toda chain can be understood in terms of a finite genus hyperelliptic curve and isospectral deformations, as van Moerbeke discovered. For instance, see [3]. So one would like to see what the relation of the Toda hierarchy is to the KdV hierarchy, how the conserved quantities are related and so forth. A start on these matters has been obtained by Toda in [4]. In this paper, the idea is that if we choose the initial data for the periodic Toda chain very carefully, then the evolution of this data under the various equations of the Toda hierarchy looks similar to the evolution of f under the KdV hierarchy. Given f, we will find a canonical choice of the initial data of the Toda equations so that the flow of this initial data under the Toda hierarchy looks like

Partially supported by NSF Grant DMS 93-05657.