Commun Math Phys 180, 505–528 (1996)

Communications in
Mathematical
Physics

On the Continuous Limit of Integrable Lattices I. The Kac–Moerbeke System and KdV Theory

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Received: 10 October 1995 / Accepted: 4 March 1996

Abstract: KdV theory is constructed systematically through the continuous limit of the Kac–Moerbeke system The infinitely many commuting vector fields, the conserved functionals, the Lax pairs and the biHamiltonian structure are recovered as the limits of suitably defined linear combinations of homologous objects for the Kac–Moerbeke system The combinatorial aspects of this recombination method are treated in detail

1. Introduction

In the literature about integrable lattices, one meets the statement that the KdV-type equations can be obtained as continuous limits of the evolution equations of suitably chosen nonlinear lattices After the pioneering work of Zabusky and Kruskal [ZK] on the continuous limit of the Fermi–Pasta–Ulam lattice, Toda pointed attention to the *integrable* lattice theories giving the KdV (or the Boussinesq) equation in the continuous limit The limit process for the Toda lattice (from now on indicated with the acronym TL) was first described by Toda and Wadati [TW], this was the beginning of an effort, aiming to get more insight on the relation between discrete integrable systems and KdV-type theories The second edition of Toda's book [Tod] quotes, for example, Saitoh's papers [Sai], other references will be given in the sequel

The discussion of the continuous limit has a counterpart in the realm of inverse scattering The papers of Case, Chiu and Kac [CCK] are a classical reference on this topic in the discrete version of inverse scattering presented here, the limit process is easily performed at any step of the construction, yielding to the standard theory for the Schrödinger spectral problem

Leaving aside the inverse scattering, and coming to the structural analysis of the integrable evolution equations, we cannot avoid quoting Kupershmidt's monograph [Kup] Here a general, purely algebraic setting is proposed for the integrable lattices, extending to discrete systems the techniques for integrable KdV-type field theories developed by the Russian school [Dic]; after constructing a formal variational