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Scaling Exponents and Multifractal Dimensions for Independent Random Cascades

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Abstract: This paper is concerned with Mandelbrot's stochastic cascade measures. The problems of (i) scaling exponents of structure functions of the measure, $\tau(q)$, and (ii) multifractal dimensions are considered for cascades with a generator vector $(w_1 \cdots w_c)$ of the general type. These problems were previously studied for independent strongly bounded variables w_i: $0 < a < w_i \leqq c$. Consequently, a broad class of models used in applications, including Kolmogorov's log-normal model in turbulence, log-stable "universal" cascades in atmospheric dynamics, has not been covered. Roughly speaking, problems (i), (ii) are here solved under the condition that the scaling exists; the τ-function is calculated for all arguments (previously this was done for positive q) and a new effect emerges: the τ-function can generally involve discontinuities in the first derivative as well as in the second.

1. Introduction and Results

Independent random cascades and cascade measures first arose as simple models of fully developed turbulence [10]. They describe the space distribution of energy dissipation rate in the transmission of energy from coarser to finer vortices. In recent years random cascades are being intensively used for theoretical interpretations of empirical data in various natural sciences [14, 15]. A gap has formed by now between theoretical models that have been investigated [1, 8] and what is of practical interest. We are going to study the multifractal characteristics of cascade measures under natural conditions of existence for these measures. This is in the first place Renyi's τ-function or scaling exponents of structure functions of different orders for a cascade measure. The function occupies a central place in the statistical theory of turbulence. The solution presented here is not complete, but the results enable one to determine $\tau(q)$, $|q| < \infty$, provided the scaling exists, and to detect a generally possible loss of continuity in the derivative $\tau(q)$ (this has not been known before). We extend Kahane's result [9] about the existence of moments of the total mass of a cascade measure to the case of negative exponents.

We begin with definitions.