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Meyer's Concept of Quasicrystal and Quasiregular Sets

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Abstract: This paper relates two mathematical concepts of long-range order of a set of atoms Λ, each of which is based on restrictions on the set of interatomic distances $\Lambda - \Lambda$. A set Λ in $\mathbb{R}^n$ is a *Meyer set* if Λ is a Delone set and there is a finite set F such that $\Lambda - \Lambda \subseteq \Lambda + F$. Y. Meyer proposed that such sets include the possible structures of quasicrystals. He obtained a structure theory for such sets, which reformulates results that he obtained in harmonic analysis around 1970, and which relates these sets to cut-and-project sets. In geometric crystallography V.I. Galiulin introduced the concept of *quasiregular set*, which is a set Λ such that both Λ and $\Lambda - \Lambda$ are Delone sets. This paper shows that these two concepts are identical.

1. Introduction

In 1984 Schechtman et al. [40] discovered materials whose X-ray diffraction spectra had sharp spots indicative of long range order, which exhibited non-crystallographic symmetries. Such materials cannot have a periodic arrangement of atoms; they are now called *quasicrystals*, cf. [27]. In the last ten years there has been an immense amount of theoretical and experimental research effort to determine the atomic structure of such materials and to find mechanisms that explain how they form and remain energetically stable, cf. [19].

On the theoretical side, a large class of structures have been constructed which exhibit sharp spots in their X-ray diffraction spectra and which have non-crystallographic symmetries. These sets consist of cut-and-project sets of points ([1], [9], [10], [23], [26], [27]). Alternate descriptions of possible quasicrystalline structures were also given in terms of quasiperiodic tilings using a finite set of prototiles ([13], [22], [29]). Many of these tiling constructions trace back to work of de Bruijn on Penrose tilings [2]. Another direction of work concerns "local rules" as analogues of short-range interactions that might explain the formation of such structures ([20], [24], [25], [28], [34], [41]). It appears that there are some quasicrystalline tilings not describable by "local rules" of particular types [4].