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Graphs and Reflection Groups

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Abstract: It is shown that graphs that generalize the ADE Dynkin diagrams and have appeared in various contexts of two-dimensional field theory may be regarded in a natural way as encoding the geometry of a root system. After recalling what are the conditions satisfied by these graphs, we define a bilinear form on a root system in terms of the adjacency matrices of these graphs and undertake the study of the group generated by the reflections in the hyperplanes orthogonal to these roots. Some "non-integrally laced" graphs are shown to be associated with subgroups of these reflection groups. The empirical relevance of these graphs in the classification of conformal field theories or in the construction of integrable lattice models is recalled, and the connections with recent developments in the context of $\mathcal{N} = 2$ supersymmetric theories and topological field theories are discussed.

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1. Introduction

Similar features have appeared recently in various problems of two-dimensional field theory and statistical mechanics. In the simplest case based on the $su(2)$ algebra, the classification of ordinary conformal field theories (cft's), of $\mathcal{N} = 2$ superconformal field theories or of the corresponding topological field theories, and the construction of integrable lattice face models have all been found to have their solutions labelled by the simply laced ADE Dynkin diagrams. This is quite remarkable in view of the fact that the setting of the problem and the techniques of analysis are in each case quite different (see [1] or Sect. 6 below for a short review and a list of references). Although our understanding of the same problems in cases of higher rank $su(N)$ is much poorer, there is some evidence that important data are again provided by a set of graphs. It is the purpose of this paper to show that these graphs may be given a geometrical interpretation as encoding the geometry (the scalar products) of a system of vectors – the "roots" – and thus enable one to construct the group generated by the reflections in the hyperplanes orthogonal to these roots.

After recalling some basic facts and introducing notations concerning reflection groups, I shall define the class of graphs that we are interested in (Sect. 2). These graphs are a natural generalization of the situation encountered with the ADE

Dedicated to: A Claude