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On the Lieb–Thirring Constants $L_{\gamma,1}$ for $\gamma \geqq 1/2$

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Abstract: Let $E_i(H)$ denote the negative eigenvalues of the one-dimensional Schrödinger operator $Hu := -u'' - Vu$, $V \geqq 0$, on $L_2(\mathbb{R})$. We prove the inequality

$$\sum_i |E_i(H)|^\gamma \leqq L_{\gamma,1} \int_{\mathbb{R}} V^{\gamma+1/2}(x)dx\,, \tag{1}$$

for the "limit" case $\gamma = 1/2$. This will imply improved estimates for the best constants $L_{\gamma,1}$ in (1) as $1/2 < \gamma < 3/2$.

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Let $H = -\Delta - V$ denote the Schrödinger operator in $L_2(\mathbb{R}^d)$. If the potential $V \geqq 0$ decreases sufficiently fast at infinity, the negative part of the spectrum of H is discrete. Let $\{E_i(H)\}$ be the corresponding increasing sequence of negative eigenvalues, each eigenvalue occurs with its multiplicity. This sequence is either finite or tends to zero.

Estimates on the behavior of the sequence of eigenvalues in terms of the potential have been in the focus of research for many years. In the earlier papers the main attention was paid to bounds on the number of negative eigenvalues ([2, 4, 18, 16, 7, 14, 12, 6]). In [15] Lieb and Thirring proved inequalities of the type

$$\sum_i |E_i(H)|^\gamma \leqq L_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma+\kappa}(x)dx, \quad \kappa = d/2\,. \tag{2}$$

Since then these estimates and the corresponding constants $L_{\gamma,d}$ have been studied intensively (e.g. [13, 9, 10]). Up to now it was known that (2) holds for all $\gamma \geqq 0$ if $d \geqq 3$, for $\gamma > 0$ if $d = 2$, and for $\gamma > 1/2$ if $d = 1$. On the contrary (2) fails for $\gamma = 0, d = 2$ and for $\gamma < 1/2, d = 1$. In this paper we prove (2) for the remaining case $d = 1, \gamma = 1/2$, which does not seem to have been settled so far. This result will imply an essential improvement for the estimates on the constants $L_{\gamma,1}$, $1/2 < \gamma < 3/2$. Moreover we deduce a new integral bound on the transmission coefficient of the corresponding scattering problem.