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A Simple Geometric Representative for μ of a Point

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Abstract: For $SU(2)$ (or $SO(3)$) Donaldson theory on a 4-manifold X, we construct a simple geometric representative for μ of a point. Let p be a generic point in X. Then the set $\{[A]|F_A^-(p)$ is reducible$\}$, with coefficient $-1/4$ and appropriate orientation, is our desired geometric representative. The construction is an exercise in real algebraic geometry in the style of Ehresmann and Pontryagin.

1. Background and Statement of Results

In the past decade, an industry has developed studying the homology of moduli spaces, thereby shedding light on the topology or geometry of underlying manifolds. The best known example is Donaldson's work on gauge theory in 4 dimensions [DK]. Donaldson's polynomial invariants measure the fundamental classes of moduli spaces of anti-self-dual connections over an orientable 4-manifold, giving information about the differentiable structure of that manifold.

Let X be an oriented 4-manifold, let $G = SU(2)$ or $SO(3)$ and let $\mathcal{B}_k$ be the space of connections (up to gauge equivalence) on P_k, the principal G bundle of instanton number k over X. Let $\mathcal{B}_k^*$ (resp. $\tilde{\mathcal{B}}_k^*$) be the space of irreducible connections, (resp. irreducible framed connections) on P_k, modulo gauge equivalence. $\tilde{\mathcal{B}}_k^*$ is a principal $SO(3)$ bundle over $\mathcal{B}_k^*$.

Donaldson [D1, D2] defined a map $\tilde{\mu} : H_i(X, \mathbb{Q}) \to H^{4-i}(\tilde{\mathcal{B}}_k^*, \mathbb{Q})$, $i = 1, 2, 3$, whose image freely generates the rational cohomology of $\tilde{\mathcal{B}}_k^*$. For Σ a 1, 2, or 3-cycle in X, the class $\tilde{\mu}([\Sigma])$ descends to a cohomology class on $\mathcal{B}_k^*$, which is then denoted $\mu([\Sigma])$. The classes $\mu([\Sigma])$, together with an additional 4-dimensional class, freely generate the cohomology of $\mathcal{B}_k^*$. The additional class can be viewed as μ of the point class $[x] \in H_0(X)$. In this view, μ maps $H_i(X)$ to $H^{4-i}(\mathcal{B}_k^*)$, where i now ranges from 0 to 3, and the image of the μ map freely generates $H^*(\mathcal{B}_k^*, \mathbb{Q})$.

This gives a polynomial invariant on the homology of X, the action of μ of the elements of H_* on the "fundamental class" of $\mathcal{M}_k$. Formally, for elements

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