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The Cohomology of the Space of Magnetic Monopoles

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Abstract: Denote by X_q the reduced space of SU_2 monopoles of charge q in $\mathbb{R}^3$. In this paper the cohomology of X_q, the cohomology with compact supports of X_q, and the image of the latter in the former are all calculated as representations of $\mathbb{Z}/q\mathbb{Z}$ which acts on X_2. This provides a non-trivial "lower bound" for the L^2 cohomology of X_q which is compatible with some conjectures of Sen. It is also shown that, granted some assumptions about the metric on X_q, its L^2 cohomology does not exceed this bound in the situation referred to in the paper as the "coprime case".

1. Introduction

The moduli space $\mathscr{M}_q$ of SU_2-monopoles of magnetic charge q in $\mathbb{R}^3$ is a Riemannian manifold of dimension $4q$. It has remarkable geometric properties, of which a comprehensive account can be found in [A-H]. Recently, to test hypotheses concerning electric-magnetic duality in non-abelian gauge theories [Sen], there has been interest in determining the square-summable harmonic forms on $\mathscr{M}_q$ – or, more precisely, on a $(4q-4)$-dimensional "reduced" moduli space X_q contained in it. To define the reduced space we first get rid of the free action of the group $\mathbb{R}^3$ of translations by restricting to monopoles whose centre of mass is at the origin in $\mathbb{R}^3$. There is still a free action of the circle group $\mathbb{T}$ which rotates the "phase" of a monopole. We cannot normalize the phase away completely, but we can fix it up to a q^{th} root of unity. This gives us a simply connected manifold X_q, on which the cyclic group μ_q of q^{th} roots of unity still acts freely by rotating the phase.

Let $\mathscr{H}_q^i$ denote the space of square-summable harmonic i-forms on X_q. We can decompose $\mathscr{H}_q^i$ according to the induced action of μ_q

$$\mathscr{H}_q^i = \bigoplus \mathscr{H}_{q,p}^i ,$$

where $\mathscr{H}_{q,p}^i$ is the part where the elements $\zeta \in \mu_q$ act by multiplication by ζ^p. Sen

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