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On Orthogonal and Symplectic Matrix Ensembles

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Abstract: The focus of this paper is on the probability, $E_\beta(0;J)$, that a set J consisting of a finite union of intervals contains no eigenvalues for the finite N Gaussian Orthogonal ($\beta = 1$) and Gaussian Symplectic ($\beta = 4$) Ensembles and their respective scaling limits both in the bulk and at the edge of the spectrum. We show how these probabilities can be expressed in terms of quantities arising in the corresponding unitary ($\beta = 2$) ensembles. Our most explicit new results concern the distribution of the largest eigenvalue in each of these ensembles. In the edge scaling limit we show that these largest eigenvalue distributions are given in terms of a particular Painlevé II function.

I. Introduction

In the standard random matrix models of $N \times N$ Hermitian or symmetric matrices the probability density that the eigenvalues lie in infinitesimal intervals about the points $x_1, \ldots, x_N$ is given by

$$P_\beta(x_1, \ldots, x_N) = C_{N\beta}\, e^{-\beta \sum V(x_i)} \prod_{j<k} |x_j - x_k|^\beta ,$$

where the constant $C_{N\beta}$ is such that the integral of the right side equals 1. In the Gaussian ensembles the potential $V(x)$ equals $x^2/2$ and the cases $\beta = 1, 2$ and 4 correspond to the orthogonal, unitary, and symplectic ensembles, respectively, since the underlying probability distributions are invariant under these groups.

When $\beta = 2$ the polynomials orthogonal with respect to the weight function $e^{-2V(x)}$ play an important role. If $\varphi_i(x)$ $(i = 0, 1, \ldots)$ is the family of functions obtained by orthonormalizing the sequence $x^i\, e^{-V(x)}$, then

$$P_2(x_1, \ldots, x_N) = \det(K_N(x_i, x_j)) \quad (i, j = 1, \ldots, N) ,$$

where

$$K_N(x, y) := \sum_{i=0}^{N-1} \varphi_i(x)\, \varphi_i(y) . \tag{1}$$