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Semiclassical Study of Quantum Scattering on the Line

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Abstract: We study the well-known problem of 1-d quantum scattering by a potential barrier in the semiclassical limit. Using the so-called exact WKB method and semiclassical microlocal analysis techniques, we get a very precise and complete description of the scattering matrix, in particular when the energy is very close to a unique, quadratic maximum of the potential. In our one-dimensional setting, we also recover the Bohr-Sommerfeld quantization condition for the resonances generated by such a maximum.

1. Introduction

This paper is devoted to the semiclassical study of quantum scattering by a potential barrier in dimension 1, in particular in the transition regime where the energy is very close to the top of the barrier. Many articles have been written since the 30's dealing with the computation of the transmission coefficient through a barrier, and this problem is one of the starting points for the development of what is nowadays called JWKB method. The exponential decay of the transmission coefficient (cf. Theorem 1 below) has been known since the first papers by R.E.Langer and H.Jeffreys (see e.g. [La] and [Je]), by the use of the famous *connection formulae*. Other techniques have been developed during the 60's, in particular by N.Fröman and P.O.Fröman (see [Fr-Fr]), M.V.Fedoryuk (see [Fe]) and F.W.J.Olver (see [Ol1]). Their works were based on a JWKB-like approximation method for the solutions of a 1-dimensional Schrödinger equation in the complex plane, often known as phase integral method, which has been recently improved by J.Ecalle and A.Voros (cf. [Ec, Vo]) and used in a new formalism by A.Grigis for the study of Hill's equation (cf. [Gr]). The new fact in what is now usually called exact-WKB analysis is that it provides, rather than approximate solutions with error bounds, exact solutions with a complete asymptotic expansion with respect to the semiclassical parameter h for example, with a priori estimates on the coefficients. A huge amount of papers has been written on this subject, and it is a difficult job to identify even the main contributions. We think that we have not forgoten too many important names, but the reader should refer to the