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Surfaces on Lie Groups, on Lie Algebras, and Their Integrability

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(with an Appendix by Juan Carlos Alvarez Paiva)

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Abstract: It is shown that the problem of the immersion of a 2-dimensional surface into a 3-dimensional Euclidean space, as well as the n-dimensional generalization of this problem, is related to the problem of studying surfaces in Lie groups and surfaces in Lie algebras. A particular case of the general formalism presented here implies that any surface can be characterized in terms of 2×2 matrices using an arbitrary parametrization. It is also shown that this generality of parametrization is useful for studying integrable surfaces, i.e. surfaces described by integrable equations. In particular starting from a suitable Lax pair (i.e. a suitable integrable equation), it is possible to construct explicitly large classes of integrable surfaces.

Introduction

Let $F = (F_1, F_2, F_3) : \pi \to \mathbb{R}^3$ be an immersion of a domain $\pi \subset \mathbb{R}^2$ into the 3-dimensional Euclidean space. Let $(u, v) \in \pi$. The Euclidean metric induces some metric $g_{11}(du)^2 + 2g_{12}dudv + g_{22}(dv)^2$ on the surface F. If this surface is sufficiently smooth, the functions $g_{ij}(u, v)$, and the functions $d_{ij}(u, v)$ which define the second fundamental form, satisfy a system of three nonlinear equations known as the Gauss–Codazzi equations. These equations are the compatibility condition of the Gauss–Weingarten equations, which are a pair of linear equations characterizing the dependence of the associated Cartan frame on u and v. There exist two geometrical characteristics on such a surface known as the Gauss curvature K and the mean curvature H.

The question of characterizing surfaces with a given K or a given H has been studied extensively both in the classic and in the recent literature. The most celebrated results in this direction correspond to constant K and to constant H. It turns out that the fundamental forms of such surfaces can be expressed in terms of solutions of the sine-Gordon and of the sinh-Gordon equations. These equations are integrable, and hence a large class of their solutions can be given explicitly. Using such global solutions, it is possible to study global properties of the associated