Commun. Math. Phys. 177, 1–11 (1996)

Communications in Mathematical Physics

On Continuity of Bowen–Ruelle–Sinai Measures in Families of One Dimensional Maps

Masato Tsujii

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ohokayama, Meguro, Tokyo 152, Japan

Received: 26 April 1993 / Accepted: 11 July 1995

Abstract: Let us consider a family of maps $Q_a(x) = ax(1-x)$ from the unit interval $[0,1]$ to itself, where $a \in [0,4]$ is the parameter. We show that, for any $\beta < 2$, there exists a subset $E \ni 4$ in $[0,4]$ with the properties

(1) $\mathrm{Leb}([4-\varepsilon, 4] - E) < \varepsilon^{\beta}$ for sufficiently small $\varepsilon > 0$,
(2) Q_a admits an absolutely continuous BRS measure μ_a when $a \in E$, and
(3) μ_a converges to the measure μ_4 as a tends to 4 on the set E.

Also we give some generalization of this results.

1. Introduction

We consider (real) one dimensional dynamical systems, that is, iterations of smooth maps f from a closed interval (or a circle) to itself. The orbit of a point x is a sequence of points

$$x, f(x), f^2(x), f^3(x), \dots .$$

In describing the distribution of the orbit, we use a sequence of probability measures

$$\mu_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}, \quad n = 1, 2, \dots ,$$

and, if this sequence converges to a probability measure μ as $n \to \infty$, we call μ the asymptotic distribution of the orbit of x. Here the convergence is that in the sense of weak topology, that is,

$$\int \varphi d\mu_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \to \int \varphi d\mu \quad \text{as } n \to \infty$$

for every continuous function φ on the interval. So the statistical properties of the orbit are given by the asymptotic distribution μ, if it exists.