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Operators with Singular Continuous Spectrum, VII. Examples with Borderline Time Decay

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Abstract: We construct one-dimensional potentials $V(x)$ so that if $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R})$, then H has purely singular spectrum; but for a dense set D, $\varphi \in D$ implies that $|(\varphi, e^{-itH}\varphi)| \leqq C_\varphi |t|^{-1/2} \ln(|t|)$ for $|t| > 2$. This implies the spectral measures have Hausdorff dimension one and also, following an idea of Malozemov–Molchanov, provides counterexamples to the direct extension of the theorem of Simon–Spencer on one-dimensional infinity high barriers.

1. Introduction

This is a continuation of my series of papers (some joint) exploring singular continuous spectrum especially in suitable Schrödinger operators and Jacobi matrices [3, 15, 4, 8, 2, 19, 17, 5, 7, 16]. Our main goal here is to construct potentials $V(x)$ on $\mathbb{R}$ so that if $H = -\frac{d^2}{dx^2} + V(x)$, then $\sigma(H) = [0, \infty)$, $\sigma_{\text{ac}}(H) = \sigma_{\text{pp}}(H) = \emptyset$, and there is a dense set $D \subset L^2(\mathbb{R})$ so that if $\varphi \in D$, then

$$|(\varphi, e^{itH}\varphi)| \leqq C_\varphi t^{-1/2} \ln(|t|) \tag{1.1}$$

for $|t| > 2$. (We say $|t| > 2$ because of the behavior of $\ln(|t|)$ for $|t| \leqq 1$; note all matrix elements are bounded by 1, so control in $|t| \leqq 2$ is trivial.)

Equation (1.1) is interesting because the stated bound on $F_\varphi(t) \equiv (\varphi, e^{-itH}\varphi)$ is just at the borderline for operators with singular continuous spectrum. Indeed, if $t^{-1/2}$ in (1.1) were replaced by $t^{-\alpha}$ for any $\alpha > \frac{1}{2}$, then $F_\varphi(t)$ would be in L^2 and so the spectral measures $d\mu_\varphi(E) = F(E)\,dE$ for $F \in L^2$; that is, $d\mu_\varphi$ would be a.c. and so $\sigma_{\text{ac}}(H) \neq \emptyset$.

As an indication of the borderline nature of (1.1), we note that by Falconer [6], (1.1) implies $d\mu_\varphi$ is a measure carried on a set of Hausdorff dimension 1 in the sense that it gives zero weight to any set of Hausdorff dimension strictly less than 1.

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