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Hidden Σ_{n+1}-Actions

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Abstract: Let n be an integer. Denote by A_n one of the following two graded vector spaces: (a) the space of all multilinear Poisson polynomials of degree n (with a grading described below), or (b) the cohomology of the space of all n-uples of complex numbers $z_1, \ldots, z_n$ with $z_i \neq z_j$ for $i \neq j$. We prove that the natural action of Σ_n on each homogeneous component of A_n can be extended to an "hidden" Σ_{n+1}-action and we compute the corresponding character (the Σ_n-character being already given by Klyaschko and Lehrer–Solomon formulas).

Introduction

Let n be an integer, let X be a symplectic manifold and let $SC_n(X)$ be the $\mathbf{Q}$-vector space generated by all multilinear maps from $(C^\infty(X))^n$ to $C^\infty(X)$ that we can obtain by composing the multiplication of functions and the Poisson bracket. It is clear that this space depends only on the dimension of X. Indeed for $\dim X \geqq (n-1)$, $SC_n(X)$ is the space of all multilinear free Poisson polynomials into n variables (see [M], Sect. 7) and it will be denoted by SC_n or by $SC_n(\infty)$. The group Σ_n acts in an obvious way on SC_n. Indeed there is a less obvious action of Σ_{n+1} on SC_n which is defined as follows. Let $p \in SC_n$ and let $w \in \Sigma_{n+1}$, where Σ_{n+1} is identified with the group of permutations of $\{0, \ldots, n\}$. There exists a unique $q \in SC_n$ such that $\int_X f_{w(0)} q(f_{w(1)}, \ldots, f_{w(n)}) = \int_X f_0 p(f_1, \ldots, f_n)$ for any compactly supported smooth functions $f_0, \ldots, f_n$ on a symplectic manifold X of dimension $\geqq n-1$, where the integral over X refers to the Liouville measure (see [M], Theorem 1.5). Then the Σ_{n+1}-action is defined by the requirement $w \cdot p = q$. This "hidden" Σ_{n+1}-action extends the natural Σ_n-action. Also the space SC_n has a natural structure of graded coalgebra ([M], Sect. 3) which is preserved by the action of the symmetric group.

Denote by U_n the space of all n-uple of complex numbers $z_1, \ldots, z_n$ with $z_i \neq z_j$ for $i \neq j$ and by SC_n^* the dual of SC_n. It turns out that the algebras $H^*(U_n)$ and SC_n^* have a very similar presentation (see [A] for the first one and [M] for the other one). Also it is natural to ask the following question: *can the natural Σ_n-action on $H^*(U_n)$ be extended to a Σ_{n+1}-action*? In this paper, we describe such an action on the cohomology with rational coefficients. However we prove that for $n \geqq 4$, no